

# Littlewood-Paley Characterizations of Hajłasz-Sobolev and Triebel-Lizorkin Spaces via Averages on Balls

Der-Chen Chang, Jun Liu, Dachun Yang\* and Wen Yuan

**Abstract** Let  $p \in (1, \infty)$  and  $q \in [1, \infty)$ . In this article, the authors characterize the Triebel-Lizorkin space  $F_{p,q}^\alpha(\mathbb{R}^n)$  with smoothness order  $\alpha \in (0, 2)$  via the Lusin-area function and the  $g_\lambda^*$ -function in terms of difference between  $f(x)$  and its average  $B_t f(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y) dy$  over a ball  $B(x,t)$  centered at  $x \in \mathbb{R}^n$  with radius  $t \in (0, 1)$ . As an application, the authors obtain a series of characterizations of  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  via pointwise inequalities, involving ball averages, in spirit close to Hajłasz gradients, here an interesting phenomena naturally appears that, in the end-point case when  $\alpha = 2$ , these pointwise inequalities characterize the Triebel-Lizorkin spaces  $F_{p,2}^2(\mathbb{R}^n)$ , while not  $F_{p,\infty}^2(\mathbb{R}^n)$ . In particular, some new pointwise characterizations of Hajłasz-Sobolev spaces via ball averages are obtained. Since these new characterizations only use ball averages, they can be used as starting points for developing a theory of Triebel-Lizorkin spaces with smoothness orders not less than 1 on spaces of homogeneous type.

## 1 Introduction

The theory of function spaces with smoothness is one of central topics of analysis on metric measure spaces. In 1996, Hajłasz [12] introduced the notion of Hajłasz gradients, which serves as a powerful tool to develop the first order Sobolev spaces on metric measure spaces. Later Shanmugalingam [23] introduced another kind of the first order Sobolev space by means of upper gradients. Via introducing the fractional version of Hajłasz gradients, Hu [17] and Yang [31] introduced Sobolev spaces with smoothness order  $\alpha \in (0, 1)$  on fractals and metric measure spaces, respectively. However, how to introduce a suitable and useful Sobolev space with smoothness order bigger than 1 on metric measure spaces is still an open problem. Due to the lack of differential structures on metric measure

---

2010 *Mathematics Subject Classification*. Primary 46E35; Secondary 42B25, 42B35, 30L99.

*Key words and phrases*. (Hajłasz-)Sobolev space, Triebel-Lizorkin space, ball average, difference, Lusin-area function,  $g_\lambda^*$ -function, Calderón reproducing formula, space of homogeneous type.

Der-Chen Chang is supported by the NSF grant DMS-1203845 and Hong Kong RGC competitive earmarked research grants #601813 and #601410. This project is also supported by the National Natural Science Foundation of China (Grant Nos. 11171027, 11361020 and 11471042), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant Nos. 2013YB60 and 2014KJJCA10).

\*Corresponding author

spaces, one key step to solve the above problem is to find some suitable substitute of the usual high order derivatives on metric measure spaces.

Via a pointwise inequality involving the higher order differences, Triebel [28, 29] and Haroske and Triebel [14, 15] obtained some pointwise characterizations, in the spirit of Hajlasz [12] (see also Hu [13] and Yang [31]), of Sobolev spaces on  $\mathbb{R}^n$  with smoothness order bigger than 1. However, it is still unclear how to introduce higher than 1 order differences on spaces of homogeneous type. Notice also that, in [22], under *a priori assumption* on the existence of polynomials, Liu et al. introduced the Sobolev spaces of higher order on metric measure spaces. Recently, Alabern et al. [1] obtained a new interesting characterization of Sobolev spaces with smoothness order bigger than 1 on  $\mathbb{R}^n$  via ball averages, which provides a possible way to introduce higher order Sobolev spaces on metric measure spaces. The corresponding characterizations for Besov and Triebel-Lizorkin spaces were later considered by Yang et al. [32].

Via differences involving ball averages, Dai et al. [7] provides several other ways, which are different from [1] and in spirit more close to the pointwise characterization as in [12, 13, 31], to introduce Sobolev spaces of order  $2\ell$  on spaces of homogeneous type in the sense of Coifman and Weiss [5, 6], where  $\ell \in \mathbb{N} := \{1, 2, \dots\}$ . Moreover, Dai et al. [8] further characterized Besov and Triebel-Lizorkin spaces with smoothness order in  $(0, 2\ell)$  via differences involving ball averages, which also gave out a possible way to introduce Besov and Triebel-Lizorkin spaces with any positive smoothness order on spaces of homogeneous type. In particular, when  $\alpha \in (0, 2)$ ,  $p \in (1, \infty)$  and  $q \in (1, \infty]$ , it was proved in [8, Theorem 3.1(ii)] that a locally integrable function  $f$  belongs to the Triebel-Lizorkin space  $F_{p,q}^\alpha(\mathbb{R}^n)$  if and only if

$$(1.1) \quad \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \sum_{k=0}^{\infty} 2^{k\alpha q} |f - B_{2^{-k}} f|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty;$$

moreover, the quantity in (1.1) is an equivalent quasi-norm of  $F_{p,q}^\alpha(\mathbb{R}^n)$ . Here and hereafter, for any locally integrable function  $f$ ,  $t \in (0, \infty)$  and  $x \in \mathbb{R}^n$ , we let

$$B_t f(x) := \frac{1}{|B(x, t)|} \int_{B(x, t)} f(y) dy =: \oint_{B(x, t)} f(y) dy,$$

and  $B(x, t)$  stand for a ball centered at  $x$  with radius  $t$ . Observe that this result in [8, Theorem 3.1(ii)] can be regarded as the characterization of  $F_{p,q}^\alpha(\mathbb{R}^n)$  via a Littlewood-Paley  $\mathcal{G}$ -function involving  $f - B_{2^{-k}} f$ . The corresponding result for homogeneous Triebel-Lizorkin spaces was also obtained in [8].

The main purpose of this article is to establish some Lusin-area function and  $g_\lambda^*$ -function variants of the above characterization for Triebel-Lizorkin spaces  $F_{p,q}^\alpha(\mathbb{R}^n)$ , which also provide some other possible ways to introduce Triebel-Lizorkin spaces with smoothness orders not less than 1 on spaces of homogeneous type. As an application, we obtain a series of characterizations of  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  for  $\alpha \in (0, 2)$  and  $p \in (1, \infty)$  via pointwise inequalities, involving ball averages, in spirit close to Hajlasz gradients, here an interesting phenomena naturally appears that, in the end-point case when  $\alpha = 2$ , these pointwise inequalities

characterize the Triebel-Lizorkin spaces  $F_{p,2}^2(\mathbb{R}^n)$ , while not  $F_{p,\infty}^2(\mathbb{R}^n)$ . Recall that, for  $p \in (1, \infty)$ , the Hajlasz-Sobolev spaces  $M^{\alpha,p}(\mathbb{R}^n)$  coincide with  $F_{p,2}^1(\mathbb{R}^n)$  when  $\alpha = 1$  and with  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  when  $\alpha \in (0, 1)$  (see [12, 31] and also Remark 1.14(i) below). Thus, these pointwise characterizations also lead to some new pointwise characterizations of (fractional) Hajlasz-Sobolev spaces in spirit of [7], which are different from those obtained in [12, 13, 17, 31]. Recall that the pointwise characterizations of Besov and Triebel-Lizorkin spaces play important and key roles in the study for the invariance of these function spaces under quasi-conformal mappings; see, for example, [20, 11, 16, 18, 2].

To state our main results of this article, we first recall some basic notions. Denote by  $L_{\text{loc}}^1(\mathbb{R}^n)$  the collection of all locally integrable functions on  $\mathbb{R}^n$ . Let  $\mathcal{S}(\mathbb{R}^n)$  denote the collection of all *Schwartz functions* on  $\mathbb{R}^n$ , endowed with the usual topology, and  $\mathcal{S}'(\mathbb{R}^n)$  its *topological dual*, namely, the collection of all bounded linear functionals on  $\mathcal{S}(\mathbb{R}^n)$  endowed with the weak \*-topology. Let  $\mathbb{Z}_+ := \{0, 1, \dots\}$  and  $\mathcal{S}_\infty(\mathbb{R}^n)$  be the set of all Schwartz functions  $\varphi$  such that  $\int_{\mathbb{R}^n} x^\gamma \varphi(x) dx = 0$  for all  $\gamma \in \mathbb{Z}_+^n$ , and  $\mathcal{S}'_\infty(\mathbb{R}^n)$  its topological dual. For all  $\alpha \in \mathbb{Z}_+^n$ ,  $m \in \mathbb{Z}_+$  and  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , let

$$\|\varphi\|_{\alpha,m} := \sup_{|\beta| \leq |\alpha|, x \in \mathbb{R}^n} (1 + |x|)^m |\partial^\beta \varphi(x)|.$$

We also use  $\widehat{\varphi} = \varphi^\wedge$  and  $\varphi^\vee$  to denote the *Fourier transform* and the *inverse transform* of  $\varphi$ , respectively. For any  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  and  $t \in (0, \infty)$ , we let  $\varphi_t(\cdot) := t^{-n} \varphi(\cdot/t)$ . For any  $E \subset \mathbb{R}^n$ , let  $\chi_E$  be its *characteristic function*.

The Triebel-Lizorkin spaces are defined as follows (see [26, 27, 10, 33]).

**Definition 1.1.** Let  $\alpha \in (0, \infty)$ ,  $p, q \in (0, \infty]$ ,  $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$  satisfy that

$$(1.2) \quad \text{supp } \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \text{ and } |\widehat{\varphi}(\xi)| \geq \text{constant} > 0 \text{ if } 3/5 \leq |\xi| \leq 5/3$$

and

$$(1.3) \quad \text{supp } \widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \text{ and } |\widehat{\Phi}(\xi)| \geq \text{constant} > 0 \text{ if } |\xi| \leq 5/3.$$

The *Triebel-Lizorkin space*  $F_{p,q}^\alpha(\mathbb{R}^n)$  is defined as the collection of all  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} < \infty$ , where, when  $p \in (0, \infty)$ ,

$$\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} := \left\| \left[ \sum_{k=0}^{\infty} 2^{k\alpha q} |\varphi_{2^{-k}} * f|^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

and

$$\|f\|_{F_{\infty,q}^\alpha(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \sup_{m \in \mathbb{Z}_+} \left\{ \int_{B(x, 2^{-m})} \sum_{k=m}^{\infty} 2^{k\alpha q} |\varphi_{2^{-k}} * f(y)|^q dy \right\}^{1/q}$$

with  $\varphi_{2^{-k}}$  when  $k = 0$  replaced by  $\Phi$  and the usual modification made when  $q = \infty$ .

**Remark 1.2.** (i) It is well known that the space  $F_{p,q}^\alpha(\mathbb{R}^n)$  is independent of the choice of the pair  $(\varphi, \Phi)$  satisfying (1.2) and (1.3).

(ii) Let  $\Phi$  and  $\varphi$  be as in Definition 1.1. It is well known that, if  $p \in (0, \infty)$ ,  $q \in (0, \infty]$  and  $\alpha \in (0, \infty)$ , then, for all  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,

$$\begin{aligned} \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} &\sim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} |\varphi_t * f|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\sim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} \int_{B(\cdot, t)} |\varphi_t * f(y)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\sim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \int_0^1 t^{-\alpha q} \left[ \int_{B(\cdot, t)} |\varphi_t * f(y)| dy \right]^q \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

with equivalent positive constants being independent of  $f$ ; see, for example, [21, 30]. Indeed, the first and the second equivalences can be found in [30, Theorem 2.6], and the third one follows from a slight modification of the proof of [30, Theorem 2.6], the details being omitted.

(iii) It is known that, when  $p \in (0, \infty)$ ,  $q \in (0, \infty]$  and  $\alpha \in (n \max\{0, 1/p - 1/q\}, 1)$ , then  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$  if and only if

$$\|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} \int_{B(\cdot, t)} |f(\cdot) - f(y)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

which also serves as an equivalent quasi-norm of  $F_{p,q}^\alpha(\mathbb{R}^n)$ ; see [27, Section 3.5.3].

The following result is a slight variant of the ‘continuous’ version of [8, Theorem 3.1(ii)] when  $p \in (0, \infty)$  and  $\ell = 1$ .

**Theorem 1.3.** *Let  $p \in (1, \infty)$ ,  $q \in (1, \infty]$  and  $\alpha \in (0, 2)$ . Then  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$  if and only if  $f \in L^p(\mathbb{R}^n)$  and*

$$|||f|||_{F_{p,q}^\alpha(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} |f - B_t f|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Moreover,  $||| \cdot |||_{F_{p,q}^\alpha(\mathbb{R}^n)}$  is an equivalent norm of  $F_{p,q}^\alpha(\mathbb{R}^n)$ .

Recall that the *Hardy-Littlewood maximal operator*  $M$  is defined by setting, for all  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,

$$Mf(x) := \sup_{x \in B} \int_B |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$  containing  $x$ .

**Remark 1.4.** Let all the notation be the same as in Theorem 1.3. Then, from the boundedness of the Hardy-Littlewood maximal function  $M$  on  $L^p(\mathbb{R}^n)$  with  $p \in (1, \infty)$ , it is easy to deduce that there exists a positive constant  $C$  such that, for all  $p \in (1, \infty)$ ,  $q \in (1, \infty]$ ,  $\alpha \in (0, 2)$  and  $f \in L^p(\mathbb{R}^n)$ ,

$$\left\| \left[ \int_1^\infty t^{-\alpha q} |f - B_t f|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \|Mf\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

By this, we conclude that

$$|||f|||_{F_{p,q}^\alpha(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^\infty t^{-\alpha q} |f - B_t f|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

with the equivalent positive constants independent of  $f$ . This further indicates that Theorem 1.3 is a natural generalization of [1, Theorem 1] in the sense that [1, Theorem 1] coincides with Theorem 1.3 in the case  $\alpha = 1$  and  $q = 2$ . We also point out that the method used to show Theorem 1.3 is similar to the proof of [8, Theorem 3.1(ii)], but totally different from the proof of [1, Theorem 1].

The main results of this article are the following characterizations of  $F_{p,q}^\alpha(\mathbb{R}^n)$  via Lusin-area functions (Theorems 1.5 and 1.6) and  $g_\lambda^*$ -functions (Theorem 1.8).

**Theorem 1.5.** *Let  $p \in (1, \infty)$ ,  $q \in (1, \infty]$ ,  $r \in [1, q)$  and  $\alpha \in (0, 2)$ . Then the following statements are equivalent:*

- (i)  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$ ;
- (ii)  $f \in L^p(\mathbb{R}^n)$  and

$$\begin{aligned} |||f|||_{F_{p,q}^\alpha(\mathbb{R}^n)}^{(r)} &:= \|f\|_{L^p(\mathbb{R}^n)} \\ &+ \left\| \left\{ \int_0^1 t^{-\alpha q} \left[ \oint_{B(\cdot, t)} |f(y) - B_t f(y)|^r dy \right]^{\frac{q}{r}} \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty. \end{aligned}$$

Moreover,  $||| \cdot |||_{F_{p,q}^\alpha(\mathbb{R}^n)}^{(r)}$  is an equivalent norm of  $F_{p,q}^\alpha(\mathbb{R}^n)$ .

For the case  $r = q$ , we have the following conclusions.

**Theorem 1.6.** *Let  $p \in (1, \infty)$  and  $q \in (1, \infty]$ .*

- (i) *If  $p \in [q, \infty)$  and  $\alpha \in (0, 2)$ , or  $p \in (1, q)$  and  $\alpha \in (n(1/p - 1/q), 1)$ , then  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$  implies that  $f \in L^p(\mathbb{R}^n)$  and*

$$\widetilde{|||f|||}_{F_{p,q}^\alpha(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} \oint_{B(\cdot, t)} |f(y) - B_t f(y)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

*is controlled by  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}$  modulus a positive constant independent of  $f$ .*

- (ii) *If  $\alpha \in (0, 2)$ , then  $f \in L^p(\mathbb{R}^n)$  and  $\widetilde{|||f|||}_{F_{p,q}^\alpha(\mathbb{R}^n)} < \infty$  imply that  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$  and  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} \leq C \widetilde{|||f|||}_{F_{p,q}^\alpha(\mathbb{R}^n)}$  for some positive constant  $C$  independent of  $f$ .*

**Remark 1.7.** We point out that the ball averages  $f_{B(\cdot, t)}$  in Theorems 1.5 and 1.6 can be replaced by  $f_{B(\cdot, \tilde{C}t)}$  for any fixed positive constant  $\tilde{C}$ .

**Theorem 1.8.** Let  $p, q \in (1, \infty)$ .

(i) If  $p \in [q, \infty)$  and  $\alpha \in (0, 2)$ , or  $p \in (1, q)$  and  $\alpha \in (n(1/p - 1/q), 1)$ , then  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$  implies that  $f \in L^p(\mathbb{R}^n)$  and

$$\begin{aligned} \overline{\|f\|}_{F_{p,q}^\alpha(\mathbb{R}^n)} &:= \|f\|_{L^p(\mathbb{R}^n)} \\ &+ \left\| \left[ \int_0^1 t^{-\alpha q} \int_{\mathbb{R}^n} \left( \frac{t}{t + |\cdot - y|} \right)^{\lambda n} |f(y) - B_t f(y)|^q \frac{dy dt}{t^{n+1}} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

is controlled by  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}$  modulus a positive constant independent of  $f$ , where

$$\lambda \in (q/\min\{q, p\}, \infty).$$

(ii) If  $\alpha \in (0, 2)$  and  $\lambda \in (1, \infty)$ , then  $f \in L^p(\mathbb{R}^n)$  and  $\overline{\|f\|}_{F_{p,q}^\alpha(\mathbb{R}^n)} < \infty$  imply that  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$  and  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} \leq C \overline{\|f\|}_{F_{p,q}^\alpha(\mathbb{R}^n)}$  for some positive constant  $C$  independent of  $f$ .

**Remark 1.9.** Observe that there exists a restriction  $\alpha \in (n(1/p - 1/q), 1)$  in Theorems 1.6(i) and 1.8(i) when  $p \in (1, q)$ . This restriction comes from an application of the Lusin-area characterization of  $F_{p,q}^\alpha(\mathbb{R}^n)$  involving the first order difference (see Remark 1.2(iii)) in the proofs of Theorems 1.6(i) and 1.8(i). We believe that  $n(1/p - 1/q)$  might be a reasonable lower bound of  $\alpha$  in Theorems 1.6(i) and 1.8(i). However, since we use  $f - B_t f$  instead of the forward first order difference in these two theorems, it might be possible that Theorems 1.6(i) and 1.8(i) remain true when  $p \in (1, q)$  and  $\alpha \in [1, 2)$ , which is still unclear so far.

By applying Theorems 1.3 and 1.6, we obtain the following pointwise characterizations of the space  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  with  $\alpha \in (0, 2)$  and  $p \in (1, \infty)$  via the average operator  $B_t$  in spirit close to Hajlasz gradients.

**Theorem 1.10.** Let  $\alpha \in (0, 2)$  and  $p \in (1, \infty)$ . Then the following statements are equivalent:

- (i)  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ ;
- (ii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and a positive constant  $C_0$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$|f(x) - B_t f(x)| \leq C_0 t^\alpha g(x).$$

Moreover, if  $\alpha \in (n/p, 1)$ , then either of (i) and (ii) is also equivalent to the following:

- (iii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_1, C_2$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, C_1 t)$ ,

$$|f(x) - B_t f(x)| \leq C_2 t^\alpha g(y).$$

In any one of the above cases, the function  $g$  can be chosen so that  $\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}$  is equivalent to  $\|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)}$  with equivalent positive constants independent of  $f$ .

The characterizations of  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  in (ii) and (iii) of Theorem 1.10 have several interesting variants, which are stated as follows.

**Theorem 1.11.** *Let  $p \in (1, \infty)$ .*

a) *If  $\alpha \in (n/p, 1)$ , then the following statements are equivalent:*

- (i)  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ ;
- (ii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_3, C_4$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$|f(x) - B_t f(x)| \leq C_3 t^\alpha \int_{B(x, C_4 t)} g(y) dy;$$

- (iii)  $f \in L^p(\mathbb{R}^n)$  and there exist  $q \in [1, p)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_5, C_6$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$|f(x) - B_t f(x)| \leq C_5 t^\alpha \left\{ \int_{B(x, C_6 t)} [g(y)]^q dy \right\}^{1/q}.$$

*In any one of the above cases, the function  $g$  can be chosen so that  $\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}$  is equivalent to  $\|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)}$  with equivalent positive constants independent of  $f$ .*

b) *If  $\alpha \in (0, 2)$ , then (ii) or (iii) in a) implies (i).*

We also have some integral variants of (ii) and (iii) in Theorem 1.10 as follows.

**Theorem 1.12.** *Let  $p \in (1, \infty)$ .*

a) *If  $\alpha \in (n/p, 1)$ , then the following statements are equivalent:*

- (i)  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ ;
- (ii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_7, C_8, C_9$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\sup_{y \in B(x, t)} |f(y) - B_{C_7 t} f(y)| \leq C_8 t^\alpha \int_{B(x, C_9 t)} g(y) dy;$$

- (iii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{10}, C_{11}, C_{12}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\int_{B(x, t)} |f(y) - B_{C_{10} t} f(y)| dy \leq C_{11} t^\alpha \int_{B(x, C_{12} t)} g(y) dy;$$

- (iv)  $f \in L^p(\mathbb{R}^n)$  and there exist  $r \in [1, \infty)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{13}, C_{14}, C_{15}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\left[ \int_{B(x, t)} |f(y) - B_{C_{13} t} f(y)|^r dy \right]^{\frac{1}{r}} \leq C_{14} t^\alpha \int_{B(x, C_{15} t)} g(y) dy;$$

(v)  $f \in L^p(\mathbb{R}^n)$  and there exist  $q \in [1, p)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{16}, C_{17}, C_{18}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\sup_{y \in B(x, t)} |f(y) - B_{C_{16}t}f(y)| \leq C_{17}t^\alpha \left\{ \int_{B(x, C_{18}t)} [g(y)]^q dy \right\}^{1/q};$$

(vi)  $f \in L^p(\mathbb{R}^n)$  and there exist  $q \in [1, p)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{19}, C_{20}, C_{21}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\int_{B(x, t)} |f(y) - B_{C_{19}t}f(y)| dy \leq C_{20}t^\alpha \left\{ \int_{B(x, C_{21}t)} [g(y)]^q dy \right\}^{1/q};$$

(vii)  $f \in L^p(\mathbb{R}^n)$  and there exist  $r \in [1, \infty)$ ,  $q \in [1, p)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{22}, C_{23}, C_{24}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\left[ \int_{B(x, t)} |f(y) - B_{C_{22}t}f(y)|^r dy \right]^{\frac{1}{r}} \leq C_{23}t^\alpha \left\{ \int_{B(x, C_{24}t)} [g(y)]^q dy \right\}^{1/q}.$$

In any one of the above cases, the function  $g$  can be chosen so that  $\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}$  is equivalent to  $\|f\|_{F_{p, \infty}^\alpha(\mathbb{R}^n)}$  with equivalent positive constants independent of  $f$ .

b) If  $\alpha \in (0, 2)$ , then any one of the above statements (ii) through (vii) in a) implies (i).

**Theorem 1.13.** Let  $p \in (1, \infty)$ .

a) If  $\alpha \in (n/p, 1)$ , then the following statements are equivalent:

- (i)  $f \in F_{p, \infty}^\alpha(\mathbb{R}^n)$ ;
- (ii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{25}, C_{26}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\sup_{y \in B(x, t)} |f(y) - B_{C_{25}t}f(y)| \leq C_{26}t^\alpha g(x);$$

(iii)  $f \in L^p(\mathbb{R}^n)$  and there exist  $r \in [1, \infty)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{27}, C_{28}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\left[ \int_{B(x, t)} |f(y) - B_{C_{27}t}f(y)|^r dy \right]^{\frac{1}{r}} \leq C_{28}t^\alpha g(x);$$

(iv)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{29}, C_{30}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\int_{B(x, t)} |f(y) - B_{C_{29}t}f(y)| dy \leq C_{30}t^\alpha g(x);$$

(v)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{31}, C_{32}, C_{33}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, C_{31}t)$ ,

$$\sup_{y \in B(x, t)} |f(y) - B_{C_{32}t}f(y)| \leq C_{33}t^\alpha g(y);$$



(vi)  $f \in L^p(\mathbb{R}^n)$  and there exist  $r \in [1, \infty)$ , a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{34}$ ,  $C_{35}$ ,  $C_{36}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, C_{34}t)$ ,

$$\left[ \int_{B(x,t)} |f(y) - B_{C_{35}t}f(y)|^r dy \right]^{\frac{1}{r}} \leq C_{36}t^\alpha g(y);$$

(vii)  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_{37}$ ,  $C_{38}$ ,  $C_{39}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, C_{37}t)$ ,

$$\int_{B(x,t)} |f(y) - B_{C_{38}t}f(y)| dy \leq C_{39}t^\alpha g(y).$$

In any one of the above cases, the function  $g$  can be chosen so that  $\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)}$  is equivalent to  $\|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)}$  with equivalent positive constants independent of  $f$ .

b) If  $\alpha \in (0, 2)$ , then any one of the above statements (ii) through (vii) in a) implies (i). Moreover, the statements (i), (iii) and (iv) in a) are equivalent for  $\alpha \in (0, 2)$ .

**Remark 1.14.** (i) Recall that, by [31, Corollary 1.3], [19, Corollary 1.2] and [20, Proposition 2.1] (see also [20, Remark 3.3(ii)]), for  $\alpha \in (0, 1)$  and  $p \in (\frac{n}{n+\alpha}, \infty)$ , the Triebel-Lizorkin space  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  coincides with the fractional Hajlasz-Sobolev space  $M^{\alpha,p}(\mathbb{R}^n)$ , which is defined in [31] as the collection of all functions  $f \in L^p(\mathbb{R}^n)$  such that there exist a nonnegative function  $g \in L^p(\mathbb{R}^n)$  and  $E \subset \mathbb{R}^n$  with measure zero so that

$$(1.4) \quad |f(x) - f(y)| \leq |x - y|^\alpha [g(x) + g(y)], \quad x, y \in \mathbb{R}^n \setminus E.$$

Such function  $g$  is called the  $\alpha$ -fractional Hajlasz gradient of  $f$ . The quasi-norm of  $f$  in  $M^{\alpha,p}(\mathbb{R}^n)$  is then given by  $\|f\|_{L^p(\mathbb{R}^n)} + \inf\{\|g\|_{L^p(\mathbb{R}^n)}\}$ , where the infimum is taken over all such  $\alpha$ -fractional Hajlasz gradients of  $f$ .

By the above equivalence, we see that Theorems 1.10 through 1.13 provide some new pointwise characterizations of fractional Hajlasz-Sobolev spaces  $M^{\alpha,p}(\mathbb{R}^n)$  via the differences between  $f$  and its ball average  $B_t f$ , which is different from the well-known pointwise characterization of  $M^{\alpha,p}(\mathbb{R}^n)$  via Hajlasz gradients as in (1.4).

(ii) It was proved in [7] that a locally integrable function  $f$  belongs to Sobolev space  $W^{2,p}(\mathbb{R}^n)$ , with  $p \in (1, \infty)$ , if and only if either of (ii) and (iii) of Theorems 1.10 and 1.11, or one of (ii) through (vii) of Theorems 1.12 and 1.13 holds true with  $\alpha = 2$ . Notice that  $W^{2,p}(\mathbb{R}^n) = F_{p,2}^2(\mathbb{R}^n)$  for all  $p \in (1, \infty)$ . Comparing Theorems 1.10 through 1.13 with [7, Theorems 1.1 through 1.4], we find a *jump* of the parameter  $q$  of Triebel-Lizorkin spaces  $F_{p,q}^\alpha(\mathbb{R}^n)$  when  $\alpha = 2$  and  $\alpha \in (0, 2)$  for the above pointwise characterizations. More precise, letting  $p \in (1, \infty)$ , any one of the items of Theorem 1.10(ii), and (iii) and (iv) of Theorem 1.13 when  $\alpha \in (0, 2)$  characterize  $F_{p,\infty}^\alpha(\mathbb{R}^n)$ , while, when  $\alpha = 2$ , they characterize  $F_{p,2}^2(\mathbb{R}^n)$ . This interesting phenomena also appears in the pointwise characterizations of Triebel-Lizorkin spaces  $F_{p,q}^\alpha(\mathbb{R}^n)$  via Hajlasz gradients with  $p \in (1, \infty)$  and  $q \in \{2, \infty\}$ , but  $\alpha \in (0, 1]$  (see [31]).

(iii) We point out that the discrete versions of Theorems 1.5 through 1.13, namely, the conclusions via replacing  $t$  by  $2^{-k}$  and  $\int_0^1 \cdots \frac{dt}{t}$  by  $\sum_{k \in \mathbb{Z}_+}$  in those statements of Theorems 1.5 through 1.13, are also true.

(iv) In view of (i) of this Remark, the pointwise characterizations in Theorems 1.10 through 1.13 provide some possible ways to introduce (fractional) Sobolev spaces with smoothness in  $(0, 2)$  on metric measure spaces. Indeed, we can prove that some statements of Theorems 1.10 through 1.13 are still equivalent on spaces of homogeneous type in Subsection 3.2 below.

The proofs of Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13 are presented in Section 2. The proof of Theorem 1.3 is similar to that of [8, Theorem 1.3(ii)]. We write  $f - B_t f$  as a convolution operator, then control  $f - B_t f$  by some maximal functions via calculating pointwise estimates of the related operator kernel and finally apply the Fefferman-Stein vector-valued maximal inequality (see, for example, [9]). The Calderón reproducing formula on  $\mathbb{R}^n$  also plays a key role in this proof. By means of Theorem 1.3, together with some known characterizations of  $F_{p,q}^\alpha(\mathbb{R}^n)$  via Lusin-area functions involving differences, we then prove Theorems 1.5 through 1.8. Using these characterizations in the limiting case  $q = \infty$ , in Theorems 1.3 through 1.6, of  $F_{p,q}^\alpha(\mathbb{R}^n)$ , we obtain the pointwise characterizations of  $F_{p,\infty}^\alpha(\mathbb{R}^n)$  in Theorems 1.10 through 1.13. This method is totally different from the method used in the proofs of [7, Theorems 1.1 through 1.4], which strongly depends on the behaviors of the Laplace operator on  $\mathbb{R}^n$  and is available only for Besov and Triebel-Lizorkin spaces with even smoothness orders and hence is not suitable for Theorems 1.10 through 1.13 in this article, since Theorems 1.10 through 1.13 concern Triebel-Lizorkin spaces with fractional smoothness orders.

Finally, Section 3 is devoted to some corresponding results of Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13 for Triebel-Lizorkin spaces with smoothness order bigger than 2. We also show some items in Theorems 1.10 through 1.13 are still equivalent on spaces of homogeneous type in the sense of Coifman and Weiss.

To end this section, we make some conventions on notation. We use the *symbol*  $A \lesssim B$  to denote that there exists a positive constant  $C$  such that  $A \leq C B$ . The symbol  $A \sim B$  is used as an abbreviation of  $A \lesssim B \lesssim A$ . Here and hereafter, the *symbol*  $C$  denotes a positive constant which is independent of the main parameters, but may depend on the fixed parameters  $n, \alpha, p, q, \lambda$  and also probably auxiliary functions, unless otherwise stated; its value may vary from line to line. For any  $p \in [1, \infty)$ , let  $p'$  denotes its *conjugate index*, namely,  $1/p + 1/p' = 1$ .

## 2 Proofs of Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13

First, we give the proof of Theorem 1.3. To this end, we need some technical lemmas. For all  $t \in (0, \infty)$  and  $x \in \mathbb{R}^n$ , let  $I(x) := \frac{1}{|B(0,1)|} \chi_{B(0,1)}(x)$  and  $I_t(x) := t^{-n} I(x/t)$ . Then

$$B_t f(x) = (f * I_t)(x), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty),$$

and hence

$$(B_t f)^\wedge(\xi) = \widehat{I}(t\xi) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n.$$

It is easy to check that

$$\widehat{I}(x) = \gamma_n \int_0^1 \cos(u|x|)(1-u^2)^{\frac{n-1}{2}} du, \quad x \in \mathbb{R}^n,$$

with  $\gamma_n := [\int_0^1 (1-u^2)^{\frac{n-1}{2}} du]^{-1}$  (see also [24, p. 430, Section 6.19]).

For all  $\lambda, q \in (1, \infty)$ ,  $\beta \in (0, \infty)$ , non-negative measurable functions  $F : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$  and  $x \in \mathbb{R}^n$ , define

$$\mathcal{G}(F)(x) := \left\{ \int_0^1 |F(x, t)|^q \frac{dt}{t} \right\}^{\frac{1}{q}},$$

$$\mathcal{S}_\beta(F)(x) := \left\{ \int_0^1 \int_{B(x, \beta t)} |F(y, t)|^q dy \frac{dt}{t} \right\}^{\frac{1}{q}}$$

and

$$\mathcal{G}_\lambda^*(F)(x) := \left\{ \int_0^1 \int_{\mathbb{R}^n} |F(y, t)|^q \left( \frac{t}{t + |x - y|} \right)^{\lambda n} dy \frac{dt}{t^{n+1}} \right\}^{\frac{1}{q}}.$$

We write  $\mathcal{S}(F) := \mathcal{S}_1(F)$ .

We have the following technical lemma.

**Lemma 2.1.** *Let  $\lambda, p, q, \beta \in (1, \infty)$ . Then there exists a positive constant  $C$  such that, for all measurable functions  $F$  on  $\mathbb{R}^n \times (0, \infty)$ ,*

(i) *for all  $x \in \mathbb{R}^n$ ,  $\mathcal{S}(F)(x) \leq C \mathcal{G}_\lambda^*(F)(x)$ ;*

(ii)

$$\|\mathcal{S}_\beta(F)\|_{L^p(\mathbb{R}^n)} \leq C \beta^{n(\frac{1}{\min\{p, q\}} - \frac{1}{q})} \|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)},$$

*where  $C$  is independent of  $\beta$  and  $F$ ;*

(iii) *for  $p \in [q, \infty)$ ,  $\|\mathcal{G}_\lambda^*(F)\|_{L^p(\mathbb{R}^n)} \leq C \|\mathcal{G}(F)\|_{L^p(\mathbb{R}^n)}$ .*

*Proof.* The proof of Lemma 2.1(i) is obvious. Similar to the proofs of [25, Theorem 4.4 and (4.3)], we can prove that Lemma 2.1(ii) holds true for  $p \in [q, \infty)$ . Now, we give the proof of Lemma 2.1(ii) for  $p \in (1, q)$ . To this end, For all  $\mu \in (0, \infty)$  and measurable functions  $F$  on  $\mathbb{R}^n \times (0, \infty)$ , let  $E_\mu := \{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > \mu \beta^{n/q}\}$  and

$$U_\mu := \{x \in \mathbb{R}^n : M(\chi_{E_\mu})(x) > (4\beta)^{-n}\},$$

where  $M$  denotes the Hardy-Littlewood maximal function. Then, by the weak type  $(1, 1)$  boundedness of  $M$ , we see that, for all  $\mu \in (0, \infty)$ ,

$$(2.1) \quad |U_\mu| \lesssim (4\beta)^n \|\chi_{E_\mu}\|_{L^1(\mathbb{R}^n)} \sim \beta^n |E_\mu|.$$

Let

$$\rho(y) := \inf \left\{ |y - z| : z \in U_\mu^c \right\},$$

where  $U_\mu^c := \mathbb{R}^n \setminus U_\mu$ . Then, by the Fubini theorem, it holds true that

$$\begin{aligned} (2.2) \quad & \int_{U_\mu^c} [\mathcal{S}_\beta(F)(x)]^q dx \\ &= \int_{U_\mu^c} \int_0^1 \int_{\{y \in \mathbb{R}^n : |y-x| < \beta t\}} |F(y, t)|^q (\beta t)^{-n} \frac{dy dt}{t} dx \\ &= \int_0^1 \int_{\{y \in \mathbb{R}^n : \rho(y) < \beta t\}} |F(y, t)|^q \left| U_\mu^c \cap B(y, \beta t) \right| (\beta t)^{-n} \frac{dy dt}{t}. \end{aligned}$$

If  $U_\mu^c \cap B(y, \beta t) \neq \emptyset$ , then there exists  $x_0 \in U_\mu^c \cap B(y, \beta t)$  and, by the definition of  $U_\mu$  and  $\beta \in (1, \infty)$ , we see that

$$\frac{|E_\mu \cap B(y, t)|}{|B(y, t)|} \leq \frac{\beta^n}{|B(y, \beta t)|} \int_{B(y, \beta t)} \chi_{E_\mu}(x) dx \leq \beta^n M(\chi_{E_\mu})(x_0) \leq 4^{-n},$$

which further implies that

$$(2.3) \quad \left| U_\mu^c \cap B(y, \beta t) \right| \lesssim \beta^n \frac{|B(y, t)|}{|E_\mu^c \cap B(y, t)|} \left| E_\mu^c \cap B(y, t) \right| \lesssim \beta^n \left| E_\mu^c \cap B(y, t) \right|.$$

If  $U_\mu^c \cap B(y, \beta t) = \emptyset$ , then (2.3) still holds true. Thus, from (2.2) and (2.3), it follows that

$$\begin{aligned} & \int_{U_\mu^c} [\mathcal{S}_\beta(F)(x)]^q dx \\ & \lesssim \int_0^1 \int_{\mathbb{R}^n} |F(y, t)|^q \left| E_\mu^c \cap B(y, t) \right| (\beta t)^{-n} \beta^n \frac{dy dt}{t} \\ & \lesssim \int_{E_\mu^c} \int_0^1 \int_{\{y \in \mathbb{R}^n : |y-x| < t\}} |F(y, t)|^q \frac{dy dt}{t^{n+1}} dx \sim \int_{E_\mu^c} [\mathcal{S}(F)(x)]^q dx. \end{aligned}$$

By this and (2.1), for all  $\ell \in \mathbb{Z}$ , we have

$$\begin{aligned} (2.4) \quad & \left| \left\{ x \in \mathbb{R}^n : \mathcal{S}_\beta(F)(x) > 2^\ell \right\} \right| \\ & \leq |U_{2^\ell}| + \left| U_{2^\ell}^c \cap \left\{ x \in \mathbb{R}^n : \mathcal{S}_\beta(F)(x) > 2^\ell \right\} \right| \\ & \lesssim \beta^n |E_{2^\ell}| + 2^{-q\ell} \int_{E_{2^\ell}^c} [\mathcal{S}(F)(x)]^q dx \\ & \sim \beta^n |E_{2^\ell}| + 2^{-q\ell} \int_0^{2^\ell \beta^{\frac{n}{q}}} \nu^{q-1} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > \nu\}| d\nu \\ & \lesssim \beta^n |E_{2^\ell}| + 2^{-q\ell} \sum_{m=-\infty}^{m_\ell} \int_{2^{m-1}}^{2^m} \nu^{q-1} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > \nu\}| d\nu \end{aligned}$$

$$\lesssim \beta^n |E_{2^\ell}| + 2^{-q\ell} \sum_{m=-\infty}^{m_\ell} 2^{q(m-1)} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > 2^{m-1}\}|,$$

where  $m_\ell := \ell + \lfloor \frac{n}{q} \log_2 \beta \rfloor + 1$  and  $\lfloor s \rfloor$  denotes the biggest integer which does not exceed the real number  $s$ .

Therefore, when  $p \in (1, q)$ , by (2.4) and the definition of  $E_\mu$ , we know that

$$\begin{aligned} & \|\mathcal{S}_\beta(F)\|_{L^p(\mathbb{R}^n)}^p \\ & \sim \sum_{\ell \in \mathbb{Z}} 2^{\ell p} \left| \left\{ x \in \mathbb{R}^n : \mathcal{S}_\beta(F)(x) > 2^\ell \right\} \right| \\ & \lesssim \sum_{\ell \in \mathbb{Z}} 2^{\ell p} \beta^n \left| \left\{ x \in \mathbb{R}^n : \mathcal{S}(F)(x) > 2^\ell \beta^{\frac{n}{q}} \right\} \right| \\ & \quad + \sum_{\ell \in \mathbb{Z}} 2^{\ell(p-q)} \sum_{m=-\infty}^{m_\ell} 2^{q(m-1)} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > 2^{m-1}\}| \\ & \lesssim \beta^{(1-\frac{p}{q})n} \|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)}^p \\ & \quad + \sum_{\ell \in \mathbb{Z}} 2^{(1-\gamma)p\ell} \beta^{\frac{n}{q}(q-p\gamma)} \sum_{m=-\infty}^{m_\ell} 2^{p\gamma(m-1)} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > 2^{m-1}\}| \\ & \sim \beta^{(1-\frac{p}{q})n} \|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)}^p \\ & \quad + \sum_{m \in \mathbb{Z}} \beta^{\frac{n}{q}(q-p\gamma)} 2^{p\gamma(m-1)} \sum_{\ell=\ell_m}^{\infty} 2^{(1-\gamma)p\ell} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > 2^{m-1}\}| \\ & \lesssim \beta^{(1-\frac{p}{q})n} \|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)}^p + \beta^{(1-\frac{p}{q})n} \sum_{m \in \mathbb{Z}} 2^{p(m-1)} |\{x \in \mathbb{R}^n : \mathcal{S}(F)(x) > 2^{m-1}\}| \\ & \sim \beta^{(1-\frac{p}{q})n} \|\mathcal{S}(F)\|_{L^p(\mathbb{R}^n)}^p, \end{aligned}$$

where  $\gamma \in (1, q/p)$  and  $\ell_m := m - \lfloor \frac{n}{q} \log_2 \beta \rfloor - 1$ , which finishes the proof of Lemma 2.1(ii) for  $p \in (1, q)$ .

Now, we show Lemma 2.1(iii). By the Fubini theorem, we see that, for any non-negative measurable function  $h$ ,

$$\begin{aligned} & \int_{\mathbb{R}^n} [\mathcal{G}_\lambda^*(F)(x)]^q h(x) dx \\ & = \int_{\mathbb{R}^n} \int_0^1 \int_{\mathbb{R}^n} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} |F(y, t)|^q \frac{dy dt}{t^{n+1}} h(x) dx \\ & = \int_0^1 \int_{\mathbb{R}^n} |F(y, t)|^q \int_{\mathbb{R}^n} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} \frac{1}{t^n} h(x) dx dy \frac{dt}{t} \\ & \leq \int_{\mathbb{R}^n} \int_0^1 |F(y, t)|^q \left[ \sup_{t \in (0, \infty)} \int_{\mathbb{R}^n} \frac{t^{(\lambda-1)n}}{(t+|x-y|)^{\lambda n}} h(x) dx \right] \frac{dt}{t} dy \\ & \lesssim \int_{\mathbb{R}^n} [\mathcal{G}(F)(y)]^q Mh(y) dy. \end{aligned}$$

Therefore, by  $p \geq q$  and the boundedness of  $M$  on  $L^{(p/q)'}(\mathbb{R}^n)$ , we find that

$$\begin{aligned}
\|\mathcal{G}_\lambda^*(F)\|_{L^p(\mathbb{R}^n)}^q &= \|[\mathcal{G}_\lambda^*(F)]^q\|_{L^{p/q}(\mathbb{R}^n)} \\
&= \sup_{\|h\|_{L^{(p/q)'}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} [\mathcal{G}_\lambda^*(F)(x)]^q h(x) dx \\
&\lesssim \sup_{\|h\|_{L^{(p/q)'}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} [\mathcal{G}(F)(x)]^q Mh(x) dx \\
&\lesssim \|\mathcal{G}(F)\|_{L^p(\mathbb{R}^n)}^q \sup_{\|h\|_{L^{(p/q)'}(\mathbb{R}^n)} \leq 1} \|Mh\|_{L^{(p/q)'}(\mathbb{R}^n)} \lesssim \|\mathcal{G}(F)\|_{L^p(\mathbb{R}^n)}^q,
\end{aligned}$$

which implies Lemma 2.1(iii) holds true and hence finishes the proof of Lemma 2.1.  $\square$

The following two lemmas come from [8, Lemmas 2.1 and 2.2], respectively.

**Lemma 2.2.** *For all  $x \in \mathbb{R}^n$ ,*

$$\widehat{I}(x) = 1 - A(|x|),$$

where

$$A(s) := 2\gamma_n \int_0^1 (1-u^2)^{\frac{n-1}{2}} \left(\sin \frac{us}{2}\right)^2 du, \quad s \in \mathbb{R}.$$

Furthermore,  $s^{-2}A(s)$  is a smooth function on  $\mathbb{R}$  satisfying that there exist positive constants  $c_1$  and  $c_2$  such that

$$(2.5) \quad 0 < c_1 \leq \frac{A(s)}{s^2} \leq c_2, \quad s \in (0, 4]$$

and

$$\sup_{s \in \mathbb{R}} \left| \left( \frac{d}{ds} \right)^i \left( \frac{A(s)}{s^2} \right) \right| < \infty, \quad i \in \mathbb{N}.$$

**Lemma 2.3.** *Let  $\{T_t\}_{t \in (0, \infty)}$  be a family of multiplier operators given by setting, for all  $f \in L^2(\mathbb{R}^n)$ ,*

$$(T_t f)^\wedge(\xi) := m(t\xi) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \quad t \in (0, \infty)$$

for some  $m \in L^\infty(\mathbb{R}^n)$ . If

$$\|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} + \|m\|_{L^1(\mathbb{R}^n)} \leq C_1 < \infty,$$

then there exists a positive constant  $C$  such that, for all  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,

$$\sup_{t \in (0, \infty)} |T_t f(x)| \leq CC_1 Mf(x).$$

The proof of Theorem 1.3 is similar to that of [8, Theorem 3.1], which is a ‘discrete’ version of Theorem 1.3. Observing that only a sketch of the proof of [8, Theorem 3.1] was given, for the sake of completeness, we give the proof of Theorem 1.3 here.

*Proof of Theorem 1.3.* Let  $\varphi$  and  $\Phi$  satisfy (1.2) and (1.3), respectively. Then there exist Schwartz functions  $\psi$  and  $\Psi$  satisfying (1.2) and (1.3), respectively, such that

$$\widehat{\Phi}(\xi)\widehat{\Psi}(\xi) + \int_0^1 \widehat{\varphi}(t\xi)\widehat{\psi}(t\xi) \frac{dt}{t} = 1, \quad \xi \in \mathbb{R}^n;$$

see, for example, [3, 4].

Let  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$ . Then it is well known that  $f \in L^p(\mathbb{R}^n)$  (see [26, Theorem/2.5.11]). Moreover, the equality

$$(2.6) \quad f = \Phi * \Psi * f + \int_0^1 \varphi_t * \psi_t * f \frac{dt}{t}$$

holds true both in  $L^p(\mathbb{R}^n)$  and  $\mathcal{S}'(\mathbb{R}^n)$ , due to the Calderón reproducing formula (see, for example, [3]). Now we show  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}$ , and it suffices to prove that

$$(2.7) \quad \left\| \left[ \int_0^1 t^{-\alpha q} |f - B_t f|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)},$$

since  $\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}$ .

Indeed, by (2.6), for all  $s, t \in (0, 1)$  and  $\xi \in \mathbb{R}^n$ ,

$$\begin{aligned} (f - B_s f)^\wedge(\xi) &= \widehat{\Phi}(\xi)A(s|\xi|)\widehat{f}_1(\xi) + \int_0^1 \widehat{\varphi}(t\xi)A(s|\xi|)\widehat{f}_t(\xi) \frac{dt}{t} \\ &=: (T_{s,1}f_1)^\wedge(\xi) + \int_0^1 (T_{s,t}f_t)^\wedge(\xi) \frac{dt}{t}, \end{aligned}$$

where  $T_{s,t}$  is given by

$$(2.8) \quad (T_{s,t}f_t)^\wedge(\xi) := \widehat{\varphi}(t\xi)A(s|\xi|)\widehat{f}_t(\xi), \quad t \in (0, 1), \quad \xi \in \mathbb{R}^n,$$

and

$$(T_{s,1}f_1)^\wedge(\xi) := \widehat{\Phi}(\xi)A(s|\xi|)\widehat{f}_1(\xi), \quad \xi \in \mathbb{R}^n,$$

with  $\widehat{f}_t := \widehat{\psi}(t \cdot)\widehat{f}$  and  $\widehat{f}_1 := \widehat{\Psi}(\cdot)\widehat{f}$ . Therefore,

$$(2.9) \quad f - B_s f = T_{s,1}f_1 + \int_0^1 T_{s,t}f_t \frac{dt}{t}.$$

For the integral part in (2.9), we split  $\int_0^1$  into two parts  $\int_0^s$  and  $\int_s^1$ . It is relatively easier to deal with the first part. Indeed, for  $t \in (0, s]$ , by (2.8), we find that, for all  $x \in \mathbb{R}^n$ ,

$$|T_{s,t}f_t(x)| = |(I - B_s)(f * \psi_t * \varphi_t)(x)| \lesssim M(f * \psi_t * \varphi_t)(x).$$

From this,  $\alpha \in (0, 2)$  and the Hölder inequality, we deduce that

$$(2.10) \quad \int_0^1 s^{-\alpha q} \left| \int_0^s T_{s,t}f_t \frac{dt}{t} \right|^q \frac{ds}{s} \lesssim \int_0^1 s^{-\alpha q} \left[ \int_0^s M(f * \psi_t * \varphi_t) \frac{dt}{t} \right]^q \frac{ds}{s}$$

$$\begin{aligned}
&\lesssim \int_0^1 s^{-\frac{\alpha q}{2}} \int_0^s [M(f * \psi_t * \varphi_t)]^q t^{-\frac{\alpha q}{2}} \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^1 t^{-\alpha q} [M(f * \psi_t * \varphi_t)]^q \frac{dt}{t}.
\end{aligned}$$

Now we estimate the integral  $\int_s^1$ . For all  $t \in (0, 1)$ ,  $s \in (0, t)$  and  $\xi \in \mathbb{R}^n$ , write

$$(T_{s,t}f_t)^\wedge(\xi) = \widehat{\varphi}(t\xi)A(s|\xi|)\widehat{f}_t(\xi) =: m_{s,t}(\xi)\widehat{f}_t(\xi),$$

where

$$m_{s,t}(\xi) := \widehat{\varphi}(t\xi) \frac{A(s|\xi|)}{(s|\xi|)^2} (s|\xi|)^2, \quad \xi \in \mathbb{R}^n.$$

Write  $\widetilde{m}_{s,t}(\xi) := m_{s,t}(t^{-1}\xi)$ . By Lemma 2.2, we see that, for all  $t \in (s, 1)$  and  $\xi \in \mathbb{R}^n$ ,

$$|\partial^\beta \widetilde{m}_{s,t}(\xi)| \lesssim \left(\frac{s}{t}\right)^2 \chi_{\overline{B(0,2)} \setminus B(0,1/2)}(\xi), \quad \beta \in \mathbb{Z}_+^n,$$

and thus

$$\|\widetilde{m}_{s,t}\|_{L^1(\mathbb{R}^n)} + \|\nabla^{n+1} \widetilde{m}_{s,t}\|_{L^1(\mathbb{R}^n)} \lesssim \left(\frac{s}{t}\right)^2,$$

which, together with Lemma 2.3, further implies that

$$|T_{s,t}f_t(x)| \lesssim \left(\frac{s}{t}\right)^2 Mf_t(x), \quad x \in \mathbb{R}^n.$$

By this, for  $\alpha \in (0, 2)$ , taking  $\beta := 1 - \alpha/2 > 0$ , together with the Hölder inequality, we conclude that

$$\begin{aligned}
(2.11) \quad \int_0^1 s^{-\alpha q} \left| \int_s^1 T_{s,t}f_t \frac{dt}{t} \right|^q \frac{ds}{s} &\lesssim \int_0^1 s^{(2-\alpha)q} \left[ \int_s^1 t^{-2} M(f_t) \frac{dt}{t} \right]^q \frac{ds}{s} \\
&\lesssim \int_0^1 s^{(2-\alpha-\beta)q} \int_s^1 [M(f_t)]^q t^{(\beta-2)q} \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^1 t^{(\beta-2)q} [M(f_t)]^q \left( \int_0^t s^{(2-\alpha-\beta)q} \frac{ds}{s} \right) \frac{dt}{t} \\
&\sim \int_0^1 t^{-\alpha q} [M(f_t)]^q \frac{dt}{t}.
\end{aligned}$$

For the part  $T_{s,1}f_1$  in (2.9), we make use of the idea used in the above estimate for  $\int_s^1$ , and find that

$$(2.12) \quad |T_{s,1}f_1(x)| \lesssim s^2 M(f_1)(x), \quad x \in \mathbb{R}^n.$$

Combining (2.10), (2.11) and (2.12) with (2.9), using the Fefferman-Stein vector-valued maximal inequality (see [9]), the independence of  $F_{p,q}^\alpha(\mathbb{R}^n)$  on the pair  $(\varphi, \Phi)$  (see Remark 1.2(i)) and Remark 1.2(ii), we see that

$$\left\| \left[ \int_0^1 s^{-\alpha q} |f - B_s f|^q \frac{ds}{s} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$



$$\begin{aligned}
&\lesssim \left\| \left[ \int_0^1 s^{(2-\alpha)q} \frac{ds}{s} \right]^{1/q} M(f * \Psi) \right\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \int_0^1 t^{-\alpha q} [M(f * \psi_t * \varphi_t)]^q \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\| \left\{ \int_0^1 t^{-\alpha q} [M(f * \psi_t)]^q \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \|f * \Psi\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} |f * \psi_t|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad + \left\| \left[ \int_0^1 t^{-\alpha q} |f * \psi_t * \varphi_t|^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\sim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)},
\end{aligned}$$

which proves (2.7).

To show the inverse direction, we only need to prove

$$(2.13) \quad \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 s^{-\alpha q} |f - B_s f|^q \frac{ds}{s} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

whenever  $f \in L^p(\mathbb{R}^n)$  and the right-hand side of (2.13) is finite. For this purpose, we first claim that

$$(2.14) \quad |f * \varphi_t(x)| \lesssim M(f - B_t f)(x), \quad t \in (0, 1), \quad x \in \mathbb{R}^n.$$

Indeed, we find that, for all  $t \in (0, 1)$  and  $\xi \in \mathbb{R}^n$ ,

$$(2.15) \quad (f * \varphi_t)^\wedge(\xi) = \frac{\widehat{\varphi}(t\xi)}{A(t|\xi|)} (f - B_t f)^\wedge(\xi) =: \eta(t\xi) (f - B_t f)^\wedge(\xi),$$

where  $\eta(\xi) := \frac{\widehat{\varphi}(\xi)}{A(|\xi|)}$  for all  $\xi \in \mathbb{R}^n$ , which is well defined due to (2.5). By Lemma 2.2, we see that  $\eta \in C_c^\infty(\mathbb{R}^n)$  and  $\text{supp } \eta \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ . The claim (2.14) then follows from Lemma 2.3.

On the other hand, it is easy to see that  $\|\Phi * f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ . From this, Remark 1.2(ii), (2.14) and the Fefferman-Stein vector-valued maximal inequality (see [9]), we deduce that

$$\begin{aligned}
\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} &\sim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} \int_{B(\cdot, t)} |\varphi_t * f(y)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \int_0^1 s^{-\alpha q} [M(f - B_s f)]^q \frac{ds}{s} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 s^{-\alpha q} |f - B_s f|^q \frac{ds}{s} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

$$\sim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.3.  $\square$

Now we prove Theorem 1.5.

*Proof of Theorem 1.5.* Let all notation be the same as in the proof of Theorem 1.3. We first prove (i) $\implies$ (ii). Let  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$ . By the Fefferman-Stein vector-valued maximal inequality (see [9]) and Theorem 1.3, we see that, for all  $r \in [1, q)$ ,

$$\begin{aligned} & \left\| \left\{ \int_0^1 s^{-\alpha q} \left[ \int_{B(\cdot, s)} |f - B_s f|^r \right]^{\frac{q}{r}} \frac{ds}{s} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \lesssim \left\| \left\{ \int_0^1 s^{-\alpha q} [M(|f - B_s f|^r)]^{\frac{q}{r}} \frac{ds}{s} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ & \lesssim \left\| \left\{ \int_0^1 s^{-\alpha q} |f - B_s f|^q \frac{ds}{s} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}, \end{aligned}$$

which finishes the proof of (i) $\implies$ (ii).

Conversely, we show (ii) $\implies$ (i). Since  $\eta$  in (2.15) is a Schwartz function, by (2.15), we observe that, for all  $t \in (0, 1)$  and  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} \int_{B(x,t)} |\varphi_t * f(y)| dy &= \int_{B(x,t)} |(\eta(t \cdot))^\vee * (f - B_t f)(y)| dy \\ &\lesssim \int_{\mathbb{R}^n} |(\eta(t \cdot))^\vee(z)| \int_{B(x,t)} |(f - B_t f)(y - z)| dy dz \\ &\lesssim M \left( \int_{B(\cdot, t)} |(f - B_t f)(y)| dy \right) (x). \end{aligned}$$

From this, by Remark 1.2(ii), the Fefferman-Stein vector-valued maximal inequality (see [9]) and the Hölder inequality, we find that, for all  $r \in [1, q)$ ,

$$\begin{aligned} \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} &\sim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \int_0^1 t^{-\alpha q} \left[ \int_{B(\cdot, t)} |\varphi_t * f(y)| dy \right]^q \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|\Phi * f\|_{L^p(\mathbb{R}^n)} \\ &\quad + \left\| \left\{ \int_0^1 t^{-\alpha q} \left[ M \left( \int_{B(\cdot, t)} |(f - B_t f)(y)| dy \right) \right]^q \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \int_0^1 t^{-\alpha q} \left[ \int_{B(\cdot, t)} |(f - B_t f)(y)| dy \right]^q \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

$$\lesssim \|\Phi * f\|_{L^p(\mathbb{R}^n)} + \left\| \left\{ \int_0^1 t^{-\alpha q} \left[ \oint_{B(\cdot, t)} |(f - B_t f)(y)|^r dy \right]^{\frac{q}{r}} \frac{dt}{t} \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

This finishes the proof of (ii) $\implies$ (i) and hence the proof of Theorem 1.5.  $\square$

Now, we prove Theorem 1.6.

*Proof of Theorem 1.6.* Let all notation be the same as in the proof of Theorem 1.3.

We first prove (i). If  $p \in [q, \infty)$  and  $\alpha \in (0, 2)$ , then the desired conclusion follows from Theorem 1.3, and (i) and (iii) of Lemma 2.1. Now we assume that  $p \in (1, q)$  and  $\alpha \in (n(1/p - 1/q), 1)$ . Let  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$ . Notice that, for all  $t \in (0, 1)$  and  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} \int_{B(x,t)} |f(y) - B_t f(y)|^q dy &\lesssim \int_{B(x,t)} \int_{B(y,t)} |f(y) - f(z)|^q dz dy \\ &\lesssim \int_{B(x,2t)} |f(y) - f(x)|^q dy. \end{aligned}$$

From this and Remark 1.2(ii), we deduce that

$$\begin{aligned} &\left\| \left[ \int_0^1 t^{-\alpha q} \oint_{B(\cdot, t)} |f(y) - B_t f(y)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\| \left[ \int_0^1 t^{-\alpha q} \oint_{B(\cdot, t)} |f(y) - f(\cdot)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}, \end{aligned}$$

which finishes the proof of Theorem 1.6(i).

Now we show (ii). Notice that, if  $\|\widetilde{f}\|_{F_{p,q}^\alpha(\mathbb{R}^n)} < \infty$ , then  $\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)}^{(1)} < \infty$  due to the Hölder inequality. Then, by Theorem 1.5 and the Hölder inequality, we have

$$\begin{aligned} \|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} &\lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} \left[ \oint_{B(\cdot, t)} |(f - B_t f)(y)| dy \right]^q \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \left[ \int_0^1 t^{-\alpha q} \oint_{B(\cdot, t)} |(f - B_t f)(y)|^q dy \frac{dt}{t} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\sim \|\widetilde{f}\|_{F_{p,q}^\alpha(\mathbb{R}^n)}. \end{aligned}$$

This finishes the proof of Theorem 1.6(ii) and hence the proof of Theorem 1.6.  $\square$

Now we employ Theorems 1.3 and 1.6 to prove Theorem 1.8.

*Proof of Theorem 1.8.* We first show (i). Let  $f \in F_{p,q}^\alpha(\mathbb{R}^n)$ . For the case when  $\alpha \in (n(1/p - 1/q), 1)$  and  $p \in (1, q)$ , by Lemma 2.1(ii) with

$$F := \mathcal{F}_\alpha(x, t) := \left| \frac{B_t f(x) - f(x)}{t^\alpha} \right|, \quad (x, t) \in \mathbb{R}^n \times (0, \infty),$$

we see that, for all  $\lambda \in (q/p, \infty)$  and  $x \in \mathbb{R}^n$ ,

$$\|\mathcal{S}_\beta(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim \beta^{n(\frac{1}{p}-\frac{1}{q})} \|\mathcal{S}(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)},$$

which, combined with

$$\begin{aligned} [\mathcal{G}_\lambda^*(\mathcal{F}_\alpha)]^q &= \int_0^1 t^{-\alpha q} \int_{|x-y|<t} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} |f(y) - B_t f(y)|^q \frac{dy dt}{t^{n+1}} \\ &\quad + \sum_{k=1}^{\infty} \int_0^1 \int_{2^{k-1}t \leq |x-y| < 2^k t} \cdots \\ &\leq \sum_{k=0}^{\infty} 2^{-kn(\lambda-1)} [\mathcal{S}_{2^k}(\mathcal{F}_\alpha)]^q \end{aligned}$$

and  $\lambda/q > 1/p$ , further implies that

$$\begin{aligned} \|\mathcal{G}_\lambda^*(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{k=0}^{\infty} 2^{-\frac{k}{q}n(\lambda-1)} \|\mathcal{S}_{2^k}(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{k=0}^{\infty} 2^{-\frac{k}{q}n(\lambda-1)} 2^{k(\frac{1}{p}-\frac{1}{q})n} \|\mathcal{S}(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)} \\ &\sim \|\mathcal{S}(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

By this, we see that the desired conclusion follows from Theorem 1.6. For the case when  $\alpha \in (0, 2)$  and  $p \in [q, \infty)$ , the desired conclusion in Theorem 1.8(i) follows from Lemma 2.1(iii) and Theorem 1.3.

Now we show (ii). By Lemma 2.1(i), we know that  $\mathcal{S}(\mathcal{F}_\alpha)(x) \lesssim \mathcal{G}_\lambda^*(\mathcal{F}_\alpha)(x)$  for all  $x \in \mathbb{R}^n$ . Then for  $f \in L^p(\mathbb{R}^n)$  with  $\|\mathcal{G}_\lambda^*(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)} < \infty$ , by Theorem 1.6, we see that

$$\|f\|_{F_{p,q}^\alpha(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\mathcal{S}(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\mathcal{G}_\lambda^*(\mathcal{F}_\alpha)\|_{L^p(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.8(ii) and hence the proof of Theorem 1.8.  $\square$

Now we use Theorems 1.3 and 1.6 to prove Theorem 1.10.

*Proof of Theorem 1.10. Step 1.* Let  $\alpha \in (0, 2)$  and  $p \in (1, \infty)$ . We first show (i)  $\implies$  (ii). Assume that  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ . Then, by Theorem 1.3, we have  $f \in L^p(\mathbb{R}^n)$  and

$$\|f\|_{L^p(\mathbb{R}^n)} + \left\| \sup_{t \in (0,1)} t^{-\alpha} |f - B_t f| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)} < \infty.$$

For any  $x \in \mathbb{R}^n$ , let  $g(x) := \sup_{t \in (0,1)} t^{-\alpha} |(f - B_t f)(x)|$ . Clearly, we see that  $g \in L^p(\mathbb{R}^n)$  and

$$|(f - B_t f)(x)| \leq t^\alpha g(x), \quad x \in \mathbb{R}^n.$$

Moreover,  $\|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)}$ . This proves (ii).

Next we show (ii) $\implies$ (i). Assume that  $f \in L^p(\mathbb{R}^n)$  and there exists a non-negative  $g \in L^p(\mathbb{R}^n)$  such that  $|(f - B_t f)(x)| \lesssim t^\alpha g(x)$  for all  $t \in (0,1)$  and almost every  $x \in \mathbb{R}^n$ . Thus,

$$\|f\|_{L^p(\mathbb{R}^n)} + \left\| \sup_{t \in (0,1)} t^{-\alpha} |f - B_t f| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} < \infty,$$

which, together with Theorem 1.3, implies that  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ . This finishes the proof of (i) $\iff$ (ii).

*Step 2.* Let  $\alpha \in (n/p, 1)$  and  $p \in (1, \infty)$ . We now show (i) $\implies$ (iii). Assume that  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ . Then, by Theorem 1.6, we have  $f \in L^p(\mathbb{R}^n)$  and

$$\|f\|_{L^p(\mathbb{R}^n)} + \left\| \sup_{t \in (0,1)} \sup_{x \in B(\cdot, t)} t^{-\alpha} |(f - B_t f)(x)| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)} < \infty.$$

For all  $y \in \mathbb{R}^n$ , let  $g(y) := \sup_{t \in (0,1)} \sup_{x \in B(y, t)} t^{-\alpha} |(f - B_t f)(x)|$ . Clearly,  $g \in L^p(\mathbb{R}^n)$  and, for all  $t \in (0,1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, t)$ ,

$$|(f - B_t f)(x)| \leq t^\alpha g(y).$$

Finally, we show (iii) $\implies$ (ii). Assume that  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_1, C_2$  such that  $t^{-\alpha} |(f - B_t f)(x)| \leq C_2 g(y)$  for all  $t \in (0,1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, C_1 t)$ . Therefore,

$$|(f - B_t f)(x)| \leq C_2 t^\alpha \int_{B(x, C_1 t)} g(y) dy \lesssim t^\alpha M g(x).$$

Noticing that  $g \in L^p(\mathbb{R}^n)$  implies  $Mg \in L^p(\mathbb{R}^n)$ , we see that (ii) holds true and hence the proof of Theorem 1.10 is finished.  $\square$

**Remark 2.4.** By the above proof, we know that (iii) $\implies$ (ii) holds true for all  $\alpha \in (0, 2)$ . The condition  $\alpha \in (n/p, 1)$  is only used for the proof of (i) $\implies$ (iii).

Now we prove Theorem 1.11.

*Proof of Theorem 1.11.* By the Hölder inequality, we immediately see that (ii) $\implies$ (iii) for all  $\alpha \in (0, 2)$ .

Next, we show (iii) $\implies$ (i) when  $\alpha \in (0, 2)$ . Assume that  $f \in L^p(\mathbb{R}^n)$  and there exists a non-negative  $g \in L^p(\mathbb{R}^n)$  such that, for all  $t \in (0,1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$|f(x) - B_t f(x)| \lesssim t^\alpha \left\{ \int_{B(x, C_6 t)} [g(y)]^q dy \right\}^{1/q}.$$

Then, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$|f(x) - B_t f(x)| \lesssim t^\alpha [M(g^q)(x)]^{1/q}.$$

Since  $g \in L^p(\mathbb{R}^n)$  and  $q \in [1, p)$ , it follows that  $[M(g^q)(x)]^{1/q} \in L^p(\mathbb{R}^n)$ , which, together with the equivalence between (i) and (ii) of Theorem 1.10, implies  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ . This proves (i).

Finally, we show (i)  $\implies$  (ii) when  $\alpha \in (n/p, 1)$ . Let  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ . Then, by the equivalence between (i) and (iii) of Theorem 1.10, we know that  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C_3, C_4$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, C_3 t)$ ,  $|(f - B_t f)(x)| \leq C_4 t^\alpha g(y)$ . Therefore

$$|(f - B_t f)(x)| \leq C_4 t^\alpha \inf_{y \in B(x, C_3 t)} g(y) \leq C_4 t^\alpha \int_{B(x, C_3 t)} g(y) dy.$$

This prove (ii) and hence finishes the proof of Theorem 1.11.  $\square$

Now we prove Theorem 1.12.

*Proof of Theorem 1.12.* By the Hölder inequality, we see that, for all  $\alpha \in (0, 2)$ ,

$$(ii) \implies (v) \implies (vii) \implies (vi)$$

and

$$(ii) \implies (iv) \implies (iii) \implies (vi).$$

Therefore, to complete the proof, it suffices to show (i)  $\implies$  (ii) and (vi)  $\implies$  (i).

Now we prove (vi)  $\implies$  (i) when  $\alpha \in (0, 2)$ . Assume that  $f$  satisfies (vi). Then there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and positive constants  $C$  and  $\tilde{C}$  such that, for all  $t \in (0, 1)$  and almost every  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} (2.16) \quad & \int_{B(x,t)} |f(y) - B_{Ct} f(y)| dy \\ & \lesssim t^\alpha \left\{ \int_{B(x,\tilde{C}t)} [g(y)]^q dy \right\}^{1/q} \\ & \lesssim t^\alpha [M(g^q)(x)]^{1/q}. \end{aligned}$$

Notice that  $g \in L^p(\mathbb{R}^n)$  and  $q \in [1, p)$  implies  $[M(g^q)(x)]^{1/q} \in L^p(\mathbb{R}^n)$ . From this, combined with (2.16) and Theorem 1.5, we deduce that

$$\begin{aligned} \|f\|_{F_{p,\infty}^\alpha(\mathbb{R}^n)} & \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| \sup_{t \in (0,1)} t^{-\alpha} \int_{B(\cdot,t)} |f(y) - B_{Ct} f(y)| dy \right\|_{L^p(\mathbb{R}^n)} \\ & \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \left\| [M(g^q)]^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty, \end{aligned}$$

which implies  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$  for all  $\alpha \in (0, 2)$ . This proves (i).

Finally, We prove (i) $\implies$ (ii) when  $\alpha \in (n/p, 1)$ . Let  $f \in F_{p,\infty}^\alpha(\mathbb{R}^n)$ . Then, by the equivalence between (i) and (iii) in Theorem 1.10, we see that  $f \in L^p(\mathbb{R}^n)$  and there exist a non-negative  $g \in L^p(\mathbb{R}^n)$  and a positive constants  $C$  such that, for all  $t \in (0, 1)$ , almost every  $y \in \mathbb{R}^n$  and  $z \in B(y, Ct)$ ,  $|f(y) - B_t f(y)| \lesssim t^\alpha g(z)$ . Therefore, for almost every  $x \in \mathbb{R}^n$  and  $y \in B(x, t)$

$$|f(y) - B_t f(y)| \lesssim t^\alpha \inf_{z \in B(y, Ct)} g(z) \lesssim t^\alpha \int_{B(y, Ct)} g(z) dz \lesssim t^\alpha \int_{B(x, (1+C)t)} g(z) dz.$$

Thus,

$$\sup_{y \in B(x, t)} |f(y) - B_t f(y)| \lesssim t^\alpha \int_{B(x, (1+C)t)} g(z) dz.$$

This proves (ii) and hence finishes the proof of Theorem 1.12.  $\square$

Finally we prove Theorem 1.13.

*Proof of Theorem 1.13.* By the Hölder inequality, it is easy to see that, for all  $\alpha \in (0, 2)$ , (ii) $\implies$ (iii) $\implies$ (iv) and (v) $\implies$ (vi) $\implies$ (vii).

Next we prove (iv) $\implies$ (i) and (vii) $\implies$ (i) when  $\alpha \in (0, 2)$ . If (iv) holds true, then, by Theorem 1.5, we see that (i) holds true; if (vii) holds true, then Theorem 1.12(iii) holds true, which further implies (i). On the other hand, from Theorem 1.5, we deduce that (i) implies (iii) for  $\alpha \in (0, 2)$ .

It remains to prove (i) $\implies$ (ii) and (i) $\implies$ (v) when  $\alpha \in (n/p, 1)$ . Indeed, if (i) holds true, then Theorem 1.12(ii) holds true, which further implies (ii) and (v). This finishes the proof of Theorem 1.13.  $\square$

### 3 Further Remarks

In this section, we first generalize some items of Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13 to the higher order Triebel-Lizorkin spaces with order bigger than 2. As a further application, we then prove that some items in Theorems 1.10 through 1.13 are still equivalent on spaces of homogeneous type, which can be used to define the Triebel-Lizorkin spaces on spaces of homogeneous type with the smoothness order  $\alpha \in (0, 2)$ .

#### 3.1 Higher Order Triebel-Lizorkin Spaces with Order Bigger Than 2

In this subsection, we consider the higher order counterparts of Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13, namely, the corresponding characterizations of Triebel-Lizorkin spaces  $F_{p,q}^\alpha(\mathbb{R}^n)$  with  $\ell \in \mathbb{N}$ ,  $p \in (1, \infty)$ ,  $q \in (1, \infty]$  and  $\alpha \in (0, 2\ell)$ . For this purpose, we need to replace the average operator  $B_t$  by its higher order variants. For all  $\ell \in \mathbb{N}$ ,  $t \in (0, \infty)$  and  $x \in \mathbb{R}^n$ , define the  $2\ell$ -th order average operator  $B_{\ell,t}$  by setting, for all  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,

$$B_{\ell,t} f(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt} f(x),$$

here and hereafter,  $\binom{2\ell}{\ell-j}$  denotes the *binomial coefficients*. Obviously,  $B_{1,t}f = B_tf$ . Moreover,

$$(B_{\ell,t}f)(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} (f * I_{jt})(x), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty).$$

If we replace the average operator  $B_t$  by  $B_{\ell,t}$  in Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13, then, by [8, Lemmas 2.1 and 2.2], we have the following theorem, whose proof is similar to the corresponding part of Theorems 1.3, 1.5, 1.6, 1.8 and 1.10 through 1.13, the details being omitted.

**Theorem 3.1.** *Let  $\ell \in \mathbb{N}$ ,  $p \in (1, \infty)$ ,  $q \in (1, \infty]$ ,  $t \in (0, 1)$  and  $\alpha \in (0, 2\ell)$ . Then the conclusions of Theorems 1.3 and 1.5, (ii) of Theorems 1.6 and 1.8, and the statements b) of Theorems 1.10 through 1.13 remain hold true when  $B_t$  is replaced by  $B_{\ell,t}$ .*

### 3.2 Triebel-Lizorkin Spaces on Spaces of Homogeneous Type

In this subsection,  $(X, \rho, \mu)$  always denotes a metric measure space of homogeneous type. Recall that a *quasi-metric* on a nonempty set  $X$  is a non-negative function  $\rho$  on  $X \times X$  which satisfies

- (i) for any  $x, y \in X$ ,  $\rho(x, y) = 0$  if and only if  $x = y$ ;
- (ii) for any  $x, y \in X$ ,  $\rho(x, y) = \rho(y, x)$ ;
- (iii) there exists a constant  $K \in [1, \infty)$  such that, for any  $x, y, z \in X$ ,

$$(3.1) \quad \rho(x, y) \leq K [\rho(x, z) + \rho(z, y)].$$

Let  $\rho$  be a quasi-metric on  $X$ , a triple  $(X, \rho, \mu)$  is called a *space of homogeneous type* in the sense of Coifman and Weiss [5, 6] if  $\mu$  is a regular Borel measure satisfying the following doubling condition, that is, there exists a constant  $\tilde{C} \in [1, \infty)$  such that, for all  $r \in (0, \infty)$  and  $x \in X$ ,

$$(3.2) \quad \mu(B(x, 2r)) \leq \tilde{C} \mu(B(x, r)),$$

where, for any given  $r \in (0, \infty)$  and  $x \in X$ , let

$$B(x, r) := \{y \in X : \rho(x, y) < r\}$$

be the quasi-metric ball related to  $\rho$  of radius  $r$  and centering at  $x$ .

The triple  $(X, \rho, \mu)$  is called a *metric measure space of homogeneous type* if  $K = 1$  in (3.1) in the above definition of the space of homogeneous type.

Clearly, if  $\mu$  is doubling, then, for any  $\gamma \in (0, \infty)$ , there exists a positive constant  $C_\gamma$ , which depends on  $\gamma$  and  $\tilde{C}$  in (3.2), such that, for all  $r \in (0, \infty)$  and  $x \in X$ ,

$$\mu(B(x, \gamma r)) \leq C_\gamma \mu(B(x, r)).$$

For all  $x \in X$  and  $t \in (0, \infty)$ , let  $B(x, t)$  denote a ball with center at  $x$  and radius  $t$ , and  $\oint_{B(x,t)} f(y) d\mu(y)$  denote the *integral average* of  $f \in L^1_{\text{loc}}(X)$  on the ball  $B(x, t) \subset X$ , that is,

$$B_tf(x) := \oint_{B(x,t)} f(y) d\mu(y) := \frac{1}{\mu(B(x,t))} \int_{B(x,t)} f(y) d\mu(y).$$



Then we have the following conclusions.

**Theorem 3.2.** *Let  $\alpha \in (0, 2)$ ,  $p \in (1, \infty)$  and  $f \in L^1_{\text{loc}}(X)$ . The following statements are equivalent:*

(i) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$  and  $y \in B(x, ct)$ ,*

$$\sup_{z \in B(x, t)} |f(z) - B_{Ct}f(z)| \leq \tilde{C}t^\alpha g(y);$$

(ii) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$\sup_{y \in B(x, t)} |f(y) - B_{Ct}f(y)| \leq \tilde{C}t^\alpha \int_{B(x, ct)} g(y) d\mu(y);$$

(iii) *there exist  $q \in [1, p)$ , a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$\sup_{y \in B(x, t)} |f(y) - B_{Ct}f(y)| \leq \tilde{C}t^\alpha \left\{ \int_{B(x, ct)} [g(y)]^q d\mu(y) \right\}^{1/q};$$

(iv) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$  and  $y \in B(x, ct)$ ,*

$$|f(x) - B_{Ct}f(x)| \leq \tilde{C}t^\alpha g(y);$$

(v) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$|f(x) - B_{Ct}f(x)| \leq \tilde{C}t^\alpha \int_{B(x, ct)} g(y) d\mu(y);$$

(vi) *there exist  $q \in [1, p)$ , a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$|f(x) - B_{Ct}f(x)| \leq \tilde{C}t^\alpha \left\{ \int_{B(x, ct)} [g(y)]^q d\mu(y) \right\}^{1/q}.$$

**Theorem 3.3.** *Let  $\alpha \in (0, 2)$ ,  $p \in (1, \infty)$ ,  $r \in [1, \infty)$  and  $f \in L^1_{\text{loc}}(X)$ . The following statements are equivalent:*

(i) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$\left[ \int_{B(x, t)} |f(y) - B_{Ct}f(y)|^r d\mu(y) \right]^{\frac{1}{r}} \leq \tilde{C}t^\alpha \int_{B(x, ct)} g(y) d\mu(y);$$

(ii) *there exist  $q \in [1, p)$ , a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$\left[ \int_{B(x,t)} |f(y) - B_{Ct}f(y)|^r d\mu(y) \right]^{\frac{1}{r}} \leq \tilde{C}t^\alpha \left\{ \int_{B(x,ct)} [g(y)]^q d\mu(y) \right\}^{1/q};$$

(iii) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$ ,*

$$\left[ \int_{B(x,t)} |f(y) - B_{Ct}f(y)|^r d\mu(y) \right]^{\frac{1}{r}} \leq \tilde{C}t^\alpha g(x);$$

(iv) *there exist a non-negative  $g \in L^p(X)$  and positive constants  $c, C, \tilde{C}$  such that, for all  $t \in (0, \infty)$  and almost every  $x \in X$  and  $y \in B(x, ct)$ ,*

$$\left[ \int_{B(x,t)} |f(z) - B_{Ct}f(z)|^r d\mu(z) \right]^{\frac{1}{r}} \leq \tilde{C}t^\alpha g(y).$$

The proofs of Theorems 3.2 and 3.3 are similar to those of [7, Theorems 3.5 and 3.6], respectively, the details being omitted.

**Remark 3.4.** It would be very interesting to establish the equivalence between the items of Theorem 3.2 and those of Theorem 3.3. Indeed, it is easy to see that Theorem 3.2(i) implies Theorem 3.3(iv). This means that the items of Theorem 3.2 imply those of Theorem 3.3. It is still unknown whether the items of Theorem 3.3 imply those of Theorem 3.2 or not.

## References

- [1] R. Alabern, J. Mateu and J. Verdera, A new characterization of Sobolev spaces on  $\mathbb{R}^n$ , *Math. Ann.* 354 (2012), 589-626.
- [2] M. Bonk, E. Saksman and T. Soto, Triebel-Lizorkin spaces on metric spaces via hyperbolic fillings, *arXiv*: 1411.5906.
- [3] A. Calderón, An atomic decomposition of distributions in parabolic  $H^p$  spaces, *Adv. Math.* 25 (1977), 216-225.
- [4] A. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, *Adv. Math.* 16 (1975), 1-64.
- [5] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Math. 242, Springer-Verlag, Berlin-New York, 1971.
- [6] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83 (1977), 569-645.
- [7] F. Dai, A. Gogatishvili, D. Yang and W. Yuan, Characterizations of Sobolev spaces via averages on balls, *Nonlinear Anal.* 128 (2015), 86-99.

- [8] F. Dai, A. Gogatishvili, D. Yang and W. Yuan, Characterizations of Besov and Triebel-Lizorkin spaces via averages on balls, *J. Math. Anal. Appl.* 433 (2016), 1350-1368.
- [9] C. Fefferman and E. M. Stein, Some maximal inequalities, *Amer. J. Math.* 93 (1971), 107-115.
- [10] M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.* 93 (1990), 34-170.
- [11] A. Gogatishvili, P. Koskela and Y. Zhou, Characterizations of Besov and Triebel-Lizorkin spaces on metric measure spaces, *Forum Math.* 25 (2013), 787-819.
- [12] P. Hajłasz, Sobolev spaces on an arbitrary metric spaces, *Potential Anal.* 5 (1996), 403-415.
- [13] P. Hajłasz, Sobolev spaces on metric-measure spaces, in: *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, 173-218, *Contemp. Math.*, 338, Amer. Math. Soc., Providence, RI, 2003.
- [14] D. D. Haroske and H. Triebel, Embeddings of function spaces: a criterion in terms of differences, *Complex Var. Elliptic Equ.* 56 (2011), 931-944.
- [15] D. D. Haroske and H. Triebel, Some recent developments in the theory of function spaces involving differences, *J. Fixed Point Theory Appl.* 13 (2013), 341-358.
- [16] S. Hencl and P. Koskela, Composition of quasiconformal mappings and functions in Triebel-Lizorkin spaces, *Math. Nachr.* 286 (2013), 669-678.
- [17] J. Hu, A note on Hajłasz-Sobolev spaces on fractals, *J. Math. Anal. Appl.* 280 (2003), 91-101.
- [18] H. Koch, P. Koskela, E. Saksman and T. Soto, Bounded compositions on scaling invariant Besov spaces, *J. Funct. Anal.* 266 (2014), 2765-2788.
- [19] P. Koskela, D. Yang and Y. Zhou, A characterization of Hajłasz-Sobolev and Triebel-Lizorkin spaces via grand Littlewood-Paley functions, *J. Funct. Anal.* 258 (2010), 2637-2661.
- [20] P. Koskela, D. Yang and Y. Zhou, Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, *Adv. Math.* 226 (2011), 3579-3621.
- [21] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets, *J. Fourier Anal. Appl.* 18 (2012), 1067-1111.
- [22] Y. Liu, G. Lu and R. L. Wheeden, Some equivalent definitions of high order Sobolev spaces on stratified groups and generalizations to metric spaces, *Math. Ann.* 323 (2002), 157-174.
- [23] N. Shanmugalingam, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoamericana* 16 (2000), 243-279.
- [24] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series 43, Monographs in Harmonic Analysis III, Princeton University Press, Princeton, NJ, 1993.
- [25] A. Torchinsky, *Real-variable Methods in Harmonic Analysis*, Pure and Applied Mathematics 123, Academic Press, Inc., Orlando, FL, 1986.
- [26] H. Triebel, *Theory of Function Spaces*, Birkhäuser Verlag, Basel, 1983.
- [27] H. Triebel, *Theory of Function Spaces. II*, Birkhäuser Verlag, Basel, 1992.

- [28] H. Triebel, Sobolev-Besov spaces of measurable functions, *Studia Math.* 201 (2010), 69-86.
- [29] H. Triebel, Limits of Besov norms, *Arch. Math. (Basel)* 96 (2011), 169-175.
- [30] T. Ullrich, Continuous characterizations of Besov-Lizorkin-Triebel spaces and new interpretations as coorbits, *J. Funct. Spaces Appl.* 2012, Art. ID 163213, 47 pp.
- [31] D. Yang, New characterizations of Hajłasz-Sobolev spaces on metric spaces, *Sci. China Ser. A* 46 (2003), 675-689.
- [32] D. Yang, W. Yuan and Y. Zhou, A new characterization of Triebel-Lizorkin spaces on  $\mathbb{R}^n$ , *Publ. Mat.* 57 (2013), 57-82.
- [33] W. Yuan, W. Sickel and D. Yang, Morrey and Campanato Meet Besov, Lizorkin and Triebel, *Lecture Notes in Mathematics* 2005, Springer-Verlag, Berlin, 2010, xi+281 pp.

Der-Chen Chang

Department of Mathematics and Department of Computer Science, Georgetown University, Washington D. C. 20057, U. S. A.

&

Department of Mathematics, Fu Jen Catholic University, Taipei 242, Taiwan, China

*E-mail:* chang@georgetown.edu

Jun Liu, Dachun Yang (Corresponding author) and Wen Yuan

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

*E-mails:* junliu@mail.bnu.edu.cn (J. Liu)

dcyang@bnu.edu.cn (D. Yang)

wenyuan@bnu.edu.cn (W. Yuan)